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Publication date:
1978

Citation for published version (APA):
Plasmans, J. E. J., & de Zeeuw, A. J. (1978). *Pareto optimality and incentives to cooperate in linear quadratic difference games*. (Research memorandum / Tilburg University, Department of Economics; Vol. FEW 75). Unknown Publisher.

subfaculteit der econometrie

RESEARCH MEMORANDUM

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RESEARCH MEMORANDUM

PARETO OPTIMALITY AND INCENTIVES TO COOPERATE IN LINEAR QUADRATIC DIFFERENCE GAMES.

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September 1978

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*) The research of the second author has been sponsored by the Netherlands Organization for the Advancement of Pure Research (ZWO).

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Abstract

Linked econometric models can often be viewed upon as a difference game. In this paper we study linear quadratic difference games. The Pareto optimal strategies and sufficient conditions for the Nash optimal strategy to be Pareto optimal will be given.

1. Introduction

In a previous paper*) it has been motivated why we are interested in N-person nonzero sum difference games. We assumed the system to be linear and time-invariant and the costs to be quadratic.
Formalizing this, we start with a system, given by the tuple

$$\{T,\ U_1,\ U_2,\ldots,\ U_N,\ \mathcal{U}_1,\ \mathcal{U}_2,\ldots,\ \mathcal{U}_N,\ Z,\ \mathcal{Z},\ Y,\ \mathcal{Y},\ X,\ f,\ \varphi\}$$

where

<u>time</u> $T = \{t_0,\ t_0+1,\ldots,t_f\}$,

<u>input alphabet (control space)</u> for the N players

$$U_k \subseteq \mathbb{R}^{s_k},\ k = 1,2,\ldots,N,$$

<u>inputs (control functions)</u> for the N players

$$\mathcal{U}_k = \{\text{functions } u_k(.):T\to U_k\},\ k = 1,2,\ldots,N,$$

<u>exogenous input alphabet</u> $Z \subseteq \mathbb{R}^r$,

<u>exogenous inputs</u> $\mathcal{Z} = \{\text{functions } z(.):T\to Z\}$,

<u>output alphabet</u> $Y \subseteq \mathbb{R}^m$,

<u>outputs</u> $\mathcal{Y} = \{\text{functions } y(.):\ T\to Y\}$,

<u>state space</u> $X \subseteq \mathbb{R}^n$,

<u>state transition function</u> $f(.):T \times X \times U_1 \times U_2 \times \ldots \times U_n \times Z\to X$

<u>read-out function</u> $\varphi(.)\ :\ T \times X \to Y$.

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*) Plasmans, J. (1978), Linked Econometric Models as a Differential game; Nash Optimality, <u>Research Memorandum</u>, Tilburg University.

The state equation is

$$x(t+1)-x(t)=f(t,x(t),u_1(t),u_2(t),\dots,u_N(t),z(t)),$$

$$t=t_0,t_0+1,\dots,t_f-1,$$

$$x(t_0) = x_0.$$

The output equation is

$$y(t) = \varphi(t, x(t)),\ t = t_0, t_0+1,\dots, t_f.$$

We assumed the system to be linear and time-invariant, so we can write the state equation and the output equation as follows:

$$x(t+1)-x(t)=A\,x(t)+\sum_{i=1}^{N} B_i\,u_i(t)+C\,z(t)$$

$$t=t_0,t_0+1,\dots,t_f-1,$$

$x(t_0) = x_0$

$y(t) = F\,x(t),\quad t = t_0,t_0+1,\dots,t_f;$

A: (n x n) matrix

B_i: (n x s_i) matrix, $i = 1,2,\dots,N$,

C: (n x r) matrix,

F: (m x n) matrix.

Each player wants to choose his control function or strategy such that his costs are minimized.

We assumed the costs to be quadratic and to be based upon deviations from some ideal path for both the output- and control functions, set by the player involved.

So, the cost functionals can be given by

$$J_k(x_0,u_1(.),u_2(.),\ldots,u_N(.),y(.)) = \tfrac{1}{2}[y(t_f)-\hat{y}_k(t_f)]'Q_k(t_f)[y(t_f)-\hat{y}_k(t_f)] +$$

$$+\tfrac{1}{2}\sum_{t=t_0}^{t_f-1}\{[y(t)-\hat{y}_k(t)]'Q_k(t)[y(t)-\hat{y}_k(t)] +$$

$$+\sum_{i=1}^{N}[u_i(t)-\hat{u}_i(t)]'R_{ki}(t)[u_i(t)-\hat{u}_i(t)]\},\ k=1,2,\ldots,N,$$

where

$\hat{y}_k(t) = F\,\hat{x}_k(t)$, $t=t_0,t_0+1,\ldots,t_f$, and $\hat{u}_k(t)$, $t=t_0,t_0+1,\ldots,t_f-1$, are the ideal paths for the output- and controlfunctions respectively, set by player k, $k=1,2,\ldots,N$;

$Q_k(t)$: $(m \times m)$ matrix, $t=t_0, t_0+1,\ldots,t_f$; $k=1,2,\ldots,N$;

$R_{ki}(t)$: $(s_i \times s_i)$ matrix, $i=1,2,\ldots,N$, $k=1,2,\ldots,N$, $t=t_0,t_0+1,\ldots,t_f-1$;

without loss of generality we can assume these matrices to be symmetric;

it is natural to assume that the matrices $Q_k(t)$, $k=1,2,\ldots,N$, $t=t_0,t_0+1,\ldots,t_f$, and $R_{kk}(t)$, $k=1,2,\ldots,N$, $t=t_0,t_0+1,\ldots,t_f-1$, are positive semi-definite;

to avoid singularities we assume the matrices $R_{kk}(t)$ to be positive definite;

in this paper we assume that the matrices $R_{ki}(t)$, $k\neq i$, are also positive semi-definite; this assumption implies that non-negative costs are assigned by player k, when any other player i deviates from his ideal control path.

We shall call a strategy <u>admissable</u> if it takes values in the input alphabet.

We distinguish two possible strategies:

a) <u>open loop</u> strategies, where each player determines his policy at the beginning of the planning period:

   $u_k(t) = p_k(x_0,t)$, $k = 1,2,\ldots,N$, $t = t_0,t_0+1,\ldots,t_f-1$;

b) <u>closed loop</u> strategies, where each player determines his policy at each time step, using new observations about the state of the system; in this way he is willing to react to the strategies of the other players; such a strategy is also called a state-feedback strategy:

$$u_k(t) = p_k(x(t),t),\ k = 1,2,\dots,N,\quad t = t_0,t_0+1,\dots,t_f-1.$$

Crucial is, in what sense each player cooperates with other players and what each player expects about the strategies of the other players.
In a previous paper*) the Nash concept of optimality was developed; this concept presupposes a competitive mood of play and an expectation of rational behaviour of the other players. In this paper we will at first focus on the Pareto concept of optimality, which presupposes a cooperative mood of play and afterwards we will work out sufficient conditions for the Nash equilibrium solution to be Pareto optimal. We claim that whenever a Nash optimal solution does not belong to the set of Pareto optimal solutions, at least one player is willing to coordinate the policies in order to obtain lower costs.

2. Pareto Optimality.

Definition 1.
A control N-tuple $\tilde{u}(.) = (\tilde{u}_1(.), \tilde{u}_2(.),\dots,\tilde{u}_N(.))$ is called a Pareto optimal (P.O.-) control if and only if for each admissable control N-tuple $u(.)$:

a) either all costs are the same, i.e.
$J_k(x_0,u(.),y(.)) = J_k(x_0, \tilde{u}(.), \tilde{y}(.))$ for all $k = 1,2,\dots,N$,

b) or there exists at least one $k \in \{1,2,\dots,N\}$, such that
$J_k(x_0,u(.),y(.)) > J_k(x_0,\tilde{u}(.),\tilde{y}(.))$.

In this case it is presupposed that the players cooperate, have confidence in one another and negotiate with each other.

*) See note page 3.

P.O.-solutions can be characterized as the set of non-inferior solutions; any solution, not belonging to this set, is dominated by some solution of this P.O.-set, in the sense, that at least one player can do better, while the others do not worse.

When we change a P.O.-strategy, at least one of the players will be worse off.

P.O.-solutions are not unique. There may be P.O.-solutions, that dominate Nash optimal (N.O.-) solutions. We will illustrate this by an example.

Example

Suppose we consider a two person, nonzero sum, static game, that can be represented by the following figure.

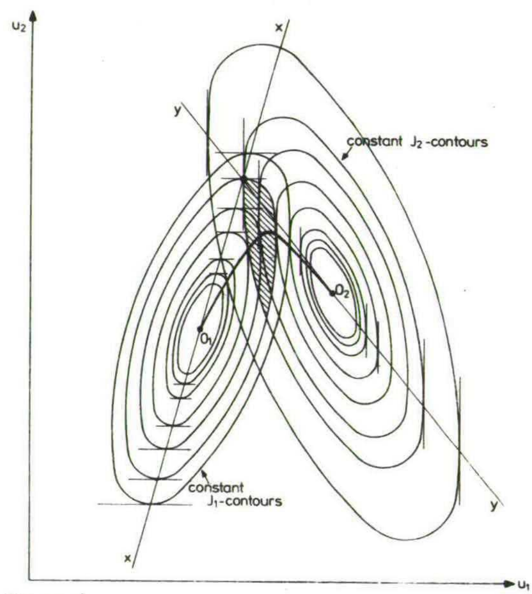

Figure a)

$J_1(u_1,u_2)$ and $J_2(u_1,u_2)$ are the cost functionals to be minimized by player 1 and 2, respectively, where u_1 and u_2 are the (scalar) decision variables. In figure a) the contours of constant costs in the u_1 u_2-space are drawn for both players. The dashed line xx represents the locus of strategies of player 1, that minimize his costs for fixed values of u_2. Similarly, yy is the locus of rational strategies of player 2 for fixed values of u_1. The intersection of these two lines is the N.O.-solution. It is easy to see from figure a), that the points within the shaded area are superior to the N.O.-solution for both players. However, cooperation of both players is absolutely necessary to reach such a situation; a solution within the shaded area cannot be arrived at by unilateral deviation of any one player from his N.O.-strategy and it is not safe against cheating by one of the players.
When the players are willing to cooperate or to negatiate, the solution set of interest becomes the set of P.O.-solutions, which is the locus of tangential points of the iso-cost contours, determined by the two cost functionals, represented by the dark line joining O_1 and O_2; this is precisely the set of non-inferior points as characterized above.
What we did want to show, namely a certain suboptimality of the N.O.-solutions as compared to some P.O.-solutions, is immediately clear now from the figure.

How can we find Pareto optimal solutions?
Definition 1 implies that it is a matter of minimization of the vector valued function

$$[J_1(x_0,u(.),y(.)),\ J_2(x_0,u(.)y(.)),\ldots,J_N(x_0,u(.),y(.))]'.$$

We can transfer the whole problem to a one player, i.e. standard optimal control, problem. In case the set of all possible values of the cost tuples $(J_1,\ J_2,\ldots J_N)$ form a convex set, solving this optimal control problem will trace out all the P.O.-solutions. The following lemmas embody these sufficient (and in case of a convex set also necessary) conditions for P.O.-solutions.

Lemma 1.

A control N-tuple $\tilde{u}(.)$ is P.O. if there exists a vector $\alpha \in \mathbb{R}^N$ with $\alpha_k > 0$, $k=1,2,\dots,N$, and $\sum_{k=1}^{N} \alpha_k = 1$, such that

$$J(x_0,\tilde{u}(.),\tilde{y}(.)) := \sum_{k=1}^{N} \alpha_k J_k(x_0,\tilde{u}(.),\tilde{y}(.)) \leqq$$

$$\leqq J(x_0,u(.),y(.)) := \sum_{k=1}^{N} \alpha_k J_k(x_0,u(.),y(.))$$

for all admissable N-tuples $u(.)$.

Proof.

Consider an admissable control N-tuple $u(.)$.

The inequality of this lemma implies either

$J(x_0,\tilde{u}(.),\tilde{y}(.)) = J(x_0,u(.),y(.))$ or

$J(x_0,\tilde{u}(.),\tilde{y}(.)) < J(x_0,u(.),y(.))$.

If $J(x_0,\tilde{u}(.),\tilde{y}(.)) = J(x_0,u(.),y(.))$ then either

$J_k(x_0,\tilde{u}(.),\tilde{y}(.)) = J_k(x_0,u(.),y(.))$ for all $k \in \{1,2,\dots,N\}$ or there is at least one $k \in \{1,2,\dots,N\}$, such that $J_k(x_0,\tilde{u}(.),\tilde{y}(.)) < J_k(x_0,u(.),y(.))$.

If $J(x_0,\tilde{u}(.),\tilde{y}(.)) < J(x_0,u(.),y(.))$ then there is at least one $k \in \{1,2,\dots,N\}$, such that

$J_k(x_0,\tilde{u}(.),\tilde{y}(.)) < J_k(x_0,u(.),y(.))$.

According to definition 1, $\tilde{u}(.)$ is a P.O.-solution.

Q.E.D.

A slightly different version of lemma 1 is:

Lemma 1'.

A control N-tuple $\tilde{u}(.)$ is P.O. if there exists a vector $\alpha \in \mathbb{R}^N$ with $\alpha_k \geqq 0$, $k=1,2,\ldots,N$, and $\sum_{k=1}^{N} \alpha_k = 1$, such that

$$J(x_0,\tilde{u}(.),\tilde{y}(.)) := \sum_{k=1}^{N} \alpha_k \; J_k(x_0,\tilde{u}(.),\tilde{y}(.)) <$$

$$< J(x_0,u(.),y(.)) := \sum_{k=1}^{N} \alpha_k \, J_k(x_0,u(.),y(.))$$

for all admissable N-tuples $u(.)$ with $u(.) \neq \tilde{u}(.)$ (i.e., $u(t) \neq \tilde{u}(t)$ for some $t \in \{t_0, t_0+1, \ldots, t_f-1\}$).

Proof.

Immediately clear from definition 1.

Remarks.

1) In lemma 1 the components of α must be strictly positive, but the minimum of $J(x_0,u(.),y(.))$ does not correspond to a unique solution, whereas in lemma 1' some components of α may be zero, but the minimum of $J(x_0,u(.),y(.))$ does correspond to a unique solution.
2) An admissable N-tuple, that yields the minimum of any players costs and that is unique, is P.O., because it can be made to fit lemma 1'.

Lemma 2.

Let $U_1 \times U_2 \times \ldots \times U_N$ be a convex set and let $J_1, J_2, \ldots, J_N$ be real-valued convex functions, defined on $U_1 \times U_2 \times \ldots \times U_N$. If the control N-tuple $\tilde{u}(.)$ is P.O. then there exists a vector $\alpha \in \mathbb{R}^N$ with $\alpha_k \geqq 0$, $k= 1,2,\ldots,N$, and $\sum_{k=1}^{N} \alpha_k = 1$, such that

$$J(x_0,\tilde{u}(.),\tilde{y}(.)) := \sum_{k=1}^{N} \alpha_k \; J_k(x_0,\tilde{u}(.),\tilde{y}(.)) \leqq$$

$$\leqq J(x_0,u(.),y(.)) := \sum_{k=1}^{N} \alpha_k \; J_k(x_0,u(.),y(.))$$

for all admissable N-tuples $u(.)$.

Proof.

Because $\tilde{u}(.)$ is P.O., we can conclude from definition 1 that, for all admissable N-tuples u(.), the system

$$J_k(x_0,u(.),y(.)) - J_k(x_0,\tilde{u}(.),\tilde{y}(.)) < 0, \quad k=1,2,\ldots,N,$$

does not have a solution in $U_1 \times U_2 \times \ldots \times U_N$.
It is a well known fact*) that in this case there exists a vector $\alpha \in \mathbb{R}^N$ with $\alpha_k \geqq 0$, $k=1,2,\ldots,N$, and $\sum_{k=1}^{N} \alpha_k = 1$,

such that

$$\sum_{k=1}^{N} \alpha_k \; (J_k(x_0,u(.),y(.)) - J_k(x_0,\tilde{u}(.),\tilde{y}(.))) \geqq 0$$

for all admissable N-tuples u(.).
Q.E.D.

3. Pareto optimal Strategies in Linear Quadratic Difference Games.

Because of the model-assumptions-linear system and quadratic cost functionals with positive (semi-)definite matrices-we can conclude from paragraph 2, that the set of P.O.-controls can be found by solving the following problem:

$$\underset{\{u_1(.),u_2(.),\ldots,u_N(.)\}}{\text{minimize}} \qquad J(x_0,u(.),y(.)) := \sum_{k=1}^{N} \alpha_k \, J_k(x_0,u(.),y(.))$$

where

*) See, e.g., Takayama, A. (1974), Mathematical Economics, The Dryden Press, Hinsdale, Illinois, pp. 67-68.

$$J_k(x_0,u(.),y(.)) := \tfrac{1}{2}\,[y(t_f)-\hat{y}_k(t_f)]'\,Q_k(t_f)\,[y(t_f)-\hat{y}_k(t_f)] + $$

$$+ \tfrac{1}{2}\sum_{t=t_0}^{t_f-1} \{[y(t)-\hat{y}_k(t)]'\,Q_k(t)[y(t)-\hat{y}_k(t)] + $$

$$+ \sum_{i=1}^{N} [u_i(t)-\hat{u}_i(t)]'\,R_{ki}(t)[u_i(t)-\hat{u}_i(t)]\},$$

$k = 1,2,\dots,N,$

subject to

$$x(t+1)-x(t) = A\,x(t) + \sum_{i=1}^{N} B_i\,u_i(t) + C\,z(t), \quad t=t_0,t_0+1,\dots,t_f-1,$$

$$x(t_0) = x_0$$

$$y(t) = F\,x(t), \quad t=t_0,t_0+1,\dots,t_f$$

for all $\alpha \in \mathbb{R}^N$ with $\alpha_k \geqq 0$ and $\sum_{k=1}^{N} \alpha_k = 1.$

The Pareto concept of optimality, which presupposes that all N players form a coalition, leads in fact to a one player problem, i.e., to a standard optimal control problem.
We see this clearly when we do some rewriting.
Define for $t = t_0, t_0+1, \dots, t_f-1$ and $i = 1,2,\dots, N$:

$$u(t) := \begin{bmatrix} u_1(t) \\ u_2(t) \\ \cdot \\ \cdot \\ \cdot \\ u_N(t) \end{bmatrix};\ \hat{u}(t) := \begin{bmatrix} \hat{u}_1(t) \\ \hat{u}_2(t) \\ \cdot \\ \cdot \\ \cdot \\ \hat{u}_N(t) \end{bmatrix};\ B := [B_1\ B_2\ \dots\ B_N];$$

$$R_i(t) := \begin{bmatrix} R_{i1}(t) & & & 0 \\ & R_{i2}(t) & & \\ & & \ddots & \\ 0 & & & R_{iN}(t) \end{bmatrix}$$

Using $\hat{y}_k(t) = F\,\hat{x}_k(t)$, $t = t_0, t_0+1, \ldots, t_f$, $k = 1,2,\ldots, N$, we arrive at the following problem:

Problem 1

$$\text{(1)}\quad \underset{u(.)}{\text{minimize}} \quad \tfrac{1}{2} \sum_{i=1}^{N} [x(t_f)-\hat{x}_i(t_f)]'\, F'\alpha_i\, Q_i(t_f)\, F\,[x(t_f)-\hat{x}_i(t_f)] + \sum_{t=t_0}^{t_f-1} \tfrac{1}{2} \Big\{ \sum_{i=1}^{N} [x(t)-\hat{x}_i(t)]'\, F'\alpha_i\, Q_i(t)\, F\,[x(t)-\hat{x}_i(t)] + \sum_{i=1}^{N} [u(t)-\hat{u}(t)]'\, \alpha_i\, R_i(t)\, [u(t)-\hat{u}(t)] \Big\}$$

subject to

$$x(t+1) = (I+A)x(t) + B\,u(t) + C\,z(t), \quad t = t_0, t_0+1, \ldots, t_f-1,$$
$$x(t_0) = x_0$$

We will state the solution of this problem in the form of a theorem.

Theorem 1.

Let $\sum_{i=1}^{N} \alpha_i\, R_i(t)$, $t=t_0, t_0+1, \ldots, t_f-1$, be positive definite.

The solution of problem (1) is given by

$$\tilde{u}(t) = -(\sum_{i=1}^{N} \alpha_i R_i(t) + B'L(t+1)B)^{-1}B'.$$

(2) $$.(L(t+1)((I+A)\tilde{x}(t)+B\,\hat{u}(t)+Cz(t)) + f(t+1)) + \hat{u}(t),$$

$$t = t_0, t_0+1, \ldots, t_f-1$$

where

$$\tilde{x}(t+1)=(I+A)\tilde{x}(t)+B\,\tilde{u}(t)+C\,z(t), \quad t = t_0, t_0+1, \ldots, t_f-2$$

(3)

$$\tilde{x}(t_0) = x_0$$

and

$L(t)$, $t= t_0, t_0+1, \ldots, t_f$, is the symmetric, positive semi-definite solution of the backward recursive Riccati equations

(4a) $$L(t) = F' \sum_{i=1}^{N} \alpha_i Q_i(t) F +$$

$$+(I+A)' L(t+1)(I-B(\sum_{i=1}^{N} \alpha_i R_i(t)+B'L(t+1)B)^{-1}B' L(t+1))(I+A)$$

(4b) $$L(t_f) = F' \sum_{i=1}^{N} \alpha_i Q_i(t_f) F$$

and

$f(t)$, $t=t_0, t_0+1, \ldots, t_f$, is the solution of the backward recursive tracking equations

(5a) $$f'(t) = - \sum_{i=1}^{N} \hat{x}_i(t)' F' \alpha_i Q_i(t) F +$$

$$+(f'(t+1)+[B\,\hat{u}(t) + C\,z(t)]'\,L(t+1)).$$

$$.(I-B(\sum_{i=1}^{N} \alpha_i\, R_i(t) + B'L(t+1)B)^{-1}\, B'L(t+1))(I+A)$$

(5b) $$f'(t_f) = - \sum_{i=1}^{N} \hat{x}_i(t_f)'F'\,\alpha_i\,Q_i(t_f)\,F$$

Remarks

1) By induction it is easy to see, that, whenever a solution of the Riccati equations (4a) and (4b) exists, it must be symmetric.
2) Because we assumed the matrices $R_{ki}(t)$, $t=t_0,t_0+1,\dots,t_f-1$, $i=1,2,\dots,N$, $k=1,2,\dots,N$, to be positive semi-definite and the matrices $R_{kk}(t)$, $t=t_0,t_0+1,\dots,t_f-1$, $k=1,2,\dots,N$, to be positive definite, we need for all $i \in \{1,2,\dots,N\}$ only one $R_{ki}(t)$ being positive definite with $\alpha_k > 0$, $k = 1,2,\dots,N$, to be able to conclude, that the matrices $\sum_{i=1}^{N} \alpha_i\, R_i(t)$, $t = t_0,t_0+1,\dots,t_f-1$, are positive definite.

   From this we see, by induction, that, whenever a solution of the Riccati equations (4a) and (4b) exists, it must be positive semi-definite and solving the backward recursive equations, we find, step by step, that

   $(\sum_{i=1}^{N} \alpha_i\, R_i(t) +B'L(t+1)B)^{-1}$ exists and is positive definite.

We will give two proofs of theorem 1; one is straightforward the other makes use of Pontryagin's discrete minimum principle.

Proof of theorem 1

A) For all $(n \times n)$ matrices $L(t)$ and $(n \times 1)$ vectors $f(t)$, $t = t_0, t_0+1,\dots,t_f$, we have

$$x'(t_f)L(t_f)x(t_f) - x'(t_0)L(t_0)x(t_0) =$$
$$= \sum_{t=t_0}^{t_f-1} \{x(t+1)'L(t+1)x(t+1) - x(t)'L(t)x(t)\}$$

and

$$f'(t_f)x(t_f) - f'(t_0)x(t_0) = \sum_{t=t_0}^{t_f-1} \{f'(t+1)x(t+1) - f'(t)x(t)\} .$$

So, with

$$L(t_f) := F' \sum_{i=1}^{N} \alpha_i \, Q_i(t_f) \, F$$

and

$$f(t_f) := - \sum_{i=1}^{N} F' \, \alpha_i \, Q_i(t_f) \, F \, \hat{x}_i(t_f)$$

we can rewrite the cost functional as follows:

$$\tfrac{1}{2}x'(t_0)L(t_0)x(t_0) + f'(t_0)x(t_0) + \tfrac{1}{2} \sum_{i=1}^{N} \hat{x}_i'(t_f)F'\alpha_i \, Q_i(t_f) F \, \hat{x}_i(t_f) +$$

$$\sum_{t=t_0}^{t_f-1} \{\tfrac{1}{2}x'(t+1)L(t+1)x(t+1) - \tfrac{1}{2}x'(t)L(t)x(t) + f'(t+1)x(t+1) - f'(t)x(t) +$$

$$+ \tfrac{1}{2} \, x'(t)F' \sum_{i=1}^{N} \alpha_i \, Q_i(t)F \, x(t) - \sum_{i=1}^{N} \hat{x}_i'(t)F'\alpha_i \, Q_i(t)F \, x(t) +$$

$$+ \tfrac{1}{2} \sum_{i=1}^{N} \hat{x}_i'(t)F'\alpha_i Q_i(t)F \, \hat{x}_i(t) + \tfrac{1}{2} \sum_{i=1}^{N} [u(t)-\hat{u}(t)]'\alpha_i \, R_i(t)[u(t)-\hat{u}(t)]\}.$$

Now we use the state equation

$$x(t+1) = (I+A)x(t) + B[u(t) - \hat{u}(t)] + B \, \hat{u}(t) + C \, z(t)$$

to arrive at the cost functional:

$$\tfrac{1}{2}\, x'(t_0)L(t_0)x(t_0) + f'(t_0)x(t_0) + \sum_{t=t_0}^{t_f} \tfrac{1}{2} \sum_{i=1}^{N} \hat{x}_i'(t)F'\alpha_i\, Q_i(t)F\, \hat{x}_i(t) +$$

$$+ \sum_{t=t_0}^{t_f-1} \{\tfrac{1}{2}[u(t) - \hat{u}(t)]'(\sum_{i=1}^{N} \alpha_i\, R_i(t) + B'\, L(t+1)B)\, [u(t) - \hat{u}(t)] +$$

$$+ \tfrac{1}{2}\, x'(t)((I+A)'L(t+1)(I+A) - L(t) + F' \sum_{i=1}^{N} \alpha_i\, Q_i(t)\, F)x(t) +$$

$$+ \tfrac{1}{2}[B\, \hat{u}(t) + C\, z(t)]'\, L(t+1)\, [B\, \hat{u}(t) + C\, z(t)] +$$

$$+[u(t)-\hat{u}(t)]'B'L(t+1)(I+A)x(t)+[B\, \hat{u}(t)+C\, z(t)]'\, L(t+1)(I+A)x(t) +$$

$$+[B\, \hat{u}(t)+ C\, z(t)]'\, L(t+1)\, B[u(t)-\hat{u}(t)] +$$

$$+ (f'(t+1)(I+A) - f'(t) - \sum_{i=1}^{N} \hat{x}_i'(t)F'\, \alpha_i\, Q_i(t)\, F)x(t) +$$

$$+ f'(t+1)B[u(t)-\hat{u}(t)] + f'(t+1)[B\, \hat{u}(t) + C\, z(t)]\}$$

Using the Riccati equations (4a) and the tracking equations (5a) we arrive, after some tedious arithmetic, at the cost functional:

$$\tfrac{1}{2}\, x'(t_0)L(t_0)x(t_0) + f'(t_0)x(t_0) + \sum_{t=t_0}^{t_f} \tfrac{1}{2} \sum_{i=1}^{N} \hat{x}_i'(t)F'\alpha_i\, Q_i(t)F\, \hat{x}_i(t) +$$

$$+ \sum_{t=t_0}^{t_f-1} \{\tfrac{1}{2}[B\, \hat{u}(t) + C\, z(t)]'\, (L(t+1)-L(t+1)B.$$

$$.\, (\sum_{i=1}^{N} \alpha_i\, R_i(t) + B'L(t+1)B)^{-1}\, B'L(t+1))[B\, \hat{u}(t) + C\, z(t)] +$$

$$+ f'(t+1)(I-B(\sum_{i=1}^{N} \alpha_i\, R_i(t) + B'L(t+1)B)^{-1}\, B'L(t+1))[B\, \hat{u}(t)+C\, z(t)] +$$

$$+ \tfrac{1}{2} f'(t+1)(-B(\sum_{i=1}^{N} \alpha_i R_i(t) + B'L(t+1)B)^{-1} B')f(t+1)\} +$$

$$+ \sum_{t=t_0}^{t_f-1} \tfrac{1}{2} [u(t) - \hat{u}(t) + (\sum_{i=1}^{N} \alpha_i R_i(t) + B'L(t+1)B)^{-1} B' .$$

$$.(L(t+1)((I+A)x(t) + B\,\hat{u}(t) + C\,z(t)) + f(t+1)]' .$$

$$.(\sum_{i=1}^{N} \alpha_i R_i(t) + B'L(t+1)B)\,[u(t) - \hat{u}(t) +$$

$$+(\sum_{i=1}^{N} \alpha_i R_i(t) + B'L(t+1)B)^{-1} B'(L(t+1)((I+A)x(t)+B\,\hat{u}(t)+C\,z(t))+f(t+1))]$$

So the cost functional has the form:

$$c + \sum_{t=t_0}^{t_f-1} \eta(t)'\, M(t)\eta(t)$$

where c is a constant,

$$\eta(t) := u(t) - \hat{u}(t) + M^{-1}(t)B'(L(t+1)((I+A)x(t) + B\,\hat{u}(t) + C\,z(t)) + f(t+1))$$

$$M(t) := \sum_{i=1}^{N} \alpha_i R_i(t) + B'L(t+1)B$$

According to remark 2 we know that $M(t)$, $t = t_0, t_0+1, \ldots, t_f-1$, is positive definite; hence we achieve minimal costs if we choose $\eta(t) = 0$, $t = t_0, t_0+1, \ldots, t_f-1$, or

$$\tilde{u}(t) = \hat{u}(t) - M^{-1}(t)B'(L(t+1)((I+A)\tilde{x}(t) + B\,\hat{u}(t) + C\,z(t)) + f(t+1)),$$

$t = t_0, t_0+1, \ldots, t_f-1$. Q.E.D.

B) We define the Hamiltonian function H by

$$H(t,x,u,\rho) := \tfrac{1}{2} \sum_{i=1}^{N} [x - \hat{x}_i(t)]' F' \alpha_i Q_i(t) F [x - \hat{x}_i(t)] +$$

$$+ \tfrac{1}{2} [u - \hat{u}(t)]' \sum_{i=1}^{N} \alpha_i R_i(t) [u - \hat{u}(t)] +$$

$$+ \rho'(A x + B u + C z(t))$$

According to Pontryagin's discrete minimum principle*) we have for the optimal control law $\tilde{u}(.)$, the corresponding state trajectory $\tilde{x}(.)$ and the corresponding co-state trajectory $\tilde{\rho}(.)$ the following relationships:

(6a) $$\tilde{x}(t+1) - \tilde{x}(t) = \frac{\partial H}{\partial \rho} (t, \tilde{x}(t), \tilde{u}(t), \tilde{\rho}(t+1)) =$$

$$= A \tilde{x}(t) + B \tilde{u}(t) + C z(t), \; t = t_0, t_0+1, \ldots, t_f-1,$$

(6b) $$\tilde{x}(t_0) = x_0;$$

(7a) $$\tilde{\rho}(t+1) - \tilde{\rho}(t) = - \frac{\partial H}{\partial x} (t, \tilde{x}(t), \tilde{u}(t), \tilde{\rho}(t+1)) =$$

$$= -F' \sum_{i=1}^{N} \alpha_i Q_i(t) F[\tilde{x}(t) - \hat{x}_i(t)] - A' \tilde{\rho}(t+1),$$

$$t = t_0, t_0+1, \ldots, t_f-1,$$

(7b) $$\tilde{\rho}(t_f) = \sum_{i=1}^{N} F' \alpha_i Q_i(t_f) F [\tilde{x}(t_f) - \hat{x}_i(t_f)]$$

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*) See, e.g., Plasmans, J.,(1975), Production Investment Behaviour, Tilburg University Press, Rotterdam, pp. 298-302.

(8) $$\frac{\partial H}{\partial u}(t, \tilde{x}(t), \tilde{u}(t), \tilde{\rho}(t+1)) = \sum_{i=1}^{N} \alpha_i R_i(t) [\tilde{u}(t) - \hat{u}(t)] + B'\tilde{\rho}(t+1) = 0,$$

$$t = t_0, t_0+1, \ldots, t_f-1.$$

Furthermore we postulate a linear relationship between the optimal state trajectory $\tilde{x}(.)$ and the optimal co-state trajectory $\tilde{\rho}(.)$:

(9) $$\tilde{\rho}(t) = L(t)\tilde{x}(t) + f(t), \quad t = t_0, t_0+1, \ldots, t_f \quad .$$

Combining (8) and (9) yields:

$$\tilde{u}(t) = -\Big(\sum_{i=1}^{N} \alpha_i R_i(t)\Big)^{-1} B'(L(t+1)\, \tilde{x}\, (t+1) + f(t+1)) + \hat{u}(t),$$

$$t = t_0,\ t_0+1, \ldots, t_f-1.$$

Using (6a) we get:

$$\tilde{u}(t) - \hat{u}(t) = -\Big(\sum_{i=1}^{N} \alpha_i R_i(t)\Big)^{-1} B'(L(t+1)((I+A)\ \tilde{x}(t) + B(\tilde{u}(t) - \hat{u}(t)) +$$

$$+ B\ \hat{u}(t) + C\ z(t)) + f(t+1)),$$

$$t = t_0, t_0+1, \ldots, t_f-1.$$

or

$$\tilde{u}(t) - \hat{u}(t) = -\Big(I + \Big(\sum_{i=1}^{N} \alpha_i R_i(t)\Big)^{-1} B'L(t+1)B\Big)^{-1} \Big(\sum_{i=1}^{N} \alpha_i R_i(t)\Big)^{-1} B' .$$

$$.\,(L(t+1)((I+A)\ \tilde{x}(t) + B\ \hat{u}(t) + C\ z(t)) + f(t+1)),$$

$$t = t_0, t_0+1, \ldots, t_f-1$$

or

$$\tilde{u}(t) = \hat{u}(t) - \left(\sum_{i=1}^{N} \alpha_i R_i(t) + B'L(t+1)B\right)^{-1} B' .$$

(10)

$$\cdot (L(t+1)((I+A)\tilde{x}(t) + B\hat{u}(t) + C z(t)) + f(t+1)),$$

$$t = t_0, t_0+1, \ldots, t_f-1.$$

How can we calculate $L(t)$ and $f(t)$, $t = t_0, t_o+1, \ldots, t_f$? From (9) and (7b) we have:

(11) $L(t_f) = F' \sum_{i=1}^{N} \alpha_i Q_i(t_f)F$ and $f(t_f) = -\sum_{i=1}^{N} F' \alpha_i Q_i(t_f)F \hat{x}_i(t_f)$

and substituting (9) into (7a) we get:

$$L(t+1)\tilde{x}(t+1) + f(t+1) - L(t)\tilde{x}(t) - f(t) =$$

$$= -F' \sum_{i=1}^{N} \alpha_i Q_i(t)F[\tilde{x}(t) - \hat{x}_i(t)] - A'(L(t+1)\tilde{x}(t+1) + f(t+1)),$$

$$t = t_0, t_0+1, \ldots, t_f-1$$

Using (6a) and (10) and rearranging terms, we get:

$$(I+A)'L(t+1)((I+A)\tilde{x}(t) - B\left(\sum_{i=1}^{N} \alpha_i R_i(t) + B'L(t+1)B\right)^{-1} B' .$$

$$\cdot (L(t+1)((I+A)\tilde{x}(t) + B\hat{u}(t) + C z(t)) + f(t+1)) + B\hat{u}(t) + C z(t)) +$$

$$+(I+A)'f(t+1) - L(t)\tilde{x}(t) - f(t) = -F' \sum_{i=1}^{N} \alpha_i Q_i(t)F[\tilde{x}(t) - \hat{x}_i(t)] ,$$

$$t = t_0, t_0+1, \ldots, t_f-1.$$

Grouping linear terms in $\tilde{x}(t)$ and also grouping constant terms, we observe, that this relationship is true if $L(t)$ and $f(t)$, $t = t_0, t_0+1, \ldots, t_f-1$, satisfy the following backward recursive equations:

$$(12)\quad L(t) = (I+A)'L(t+1)(I-B(\sum_{i=1}^{N} \alpha_i R_i(t) + B'L(t+1)B)^{-1} B'L(t+1))(I+A) +$$

$$+ F' \sum_{i=1}^{N} \alpha_i Q_i(t)F, \quad t = t_0, t_0+1, \ldots, t_f-1$$

and

$$(13)\quad f(t) = -F' \sum_{i=1}^{N} \alpha_i Q_i(t)F \hat{x}_i(t) + (I+A)'(I-L(t+1)B.$$

$$.(\sum_{i=1}^{N} \alpha_i R_i(t)+B'L(t+1)B)^{-1} B'(f(t+1)+L(t+1)(B \hat{u}(t)+C z(t))),$$

$$t = t_0, t_0+1, \ldots, t_f-1$$

(10) + (11) + (12) + (13): Q.E.D.

Remarks.

1) For each vector $\alpha \in \mathbb{R}^N$ with $\alpha_k \geqq 0$, $k = 1,2,\ldots,N$, and $\sum_{k=1}^{N} \alpha_k = 1$ we found a Pareto optimal feedback control law.

By solving the state equations, we can write the current state explicitly as a function of the initial state and the past values of the control functions; this yields the open loop control law. In case of one player the optimal open loop- and closed loop control laws give the same result.

2) Although proof A needs a little bit more computational effort than proof B, it has the advantage, that it is straightforward and it gives an explicit expression for the costs; we will state this again in corollary 1.

3) The Pareto optimal control law (2) can be "separated" into P.O.-control laws for the N players if $R_{ki}(t) \equiv 0$, $i \neq k$, $i = 1,2,\dots,N$, $k = 1,2,\dots,N$, $t = t_0, t_0+1,\dots,t_f-1$ and $\alpha_k > 0$ for all $k \in \{1,2,\dots,N\}$; we will show this in corollary 2.

Corollary 1.

The Pareto optimal costs for problem (1) are given by:

$$\tfrac{1}{2}\, x'(t_0)L(t_0)x(t_0)+f'(t_0)x(t_0) + \sum_{t=t_0}^{t_f} \tfrac{1}{2} \sum_{i=1}^{N} \hat{x}_i'(t)F'\alpha_i\, Q_i(t)F\, \hat{x}_i(t) +$$

$$+ \sum_{t=t_0}^{t_f-1} \{\tfrac{1}{2}\,[B\,\hat{u}(t) + C\,z(t)]'\, L(t+1)(I-B.$$

$$.(\sum_{i=1}^{N} \alpha_i\, R_i(t) + B'L(t+1)B)^{-1}\, B'L(t+1))[B\,\hat{u}(t) + C\,z(t)] +$$

$$+ f'(t+1)(I-B(\sum_{i=1}^{N} \alpha_i\, R_i(t) + B'L(t+1)B)^{-1}\, B'L(t+1))[B\,\hat{u}(t)+ C\,z(t)] +$$

$$- \tfrac{1}{2}\, f'(t+1)B(\sum_{i=1}^{N} \alpha_i\, R_i(t) + B'L(t+1)B)^{-1}\, B'f(t+1)\}$$

Proof.

Follows directly from proof A of theorem 1.

Corollary 2.

If $R_{ki}(t) \equiv 0$, $i \neq k$, $i = 1,2,\dots,N$, $k = 1,2,\dots,N$, $t = t_0, t_0+1,\dots,t_f-1$, then for each vector $\alpha \in \mathbb{R}^N$ with $\alpha_k > 0$,

$k = 1,2,\ldots,N$, and $\sum_{k=1}^{N} \alpha_k = 1$,

a P.O.-control law for player k, $k = 1,2,\ldots,N$, is given by:

$$\tilde{u}_k(t) = -\frac{1}{\alpha_k} R_{kk}^{-1}(t)B_k'L(t+1)D^{-1}(t+1)((I+A)\tilde{x}(t)+B\,\hat{u}(t) + C\,z(t)) +$$

$$-\frac{1}{\alpha_k} R_{kk}^{-1}(t)B_k'D^{-1}(t+1)f(t+1) + \hat{u}_k(t), \quad t = t_0,t_0+1,..,t_f-1,$$

where

$$D^{-1}(t+1) := I-B(\sum_{i=1}^{N} \alpha_i R_i(t)+B'L(t+1)B)^{-1} B'L(t+1), \quad t = t_0,t_0+1,\ldots,t_f-1.$$

<u>Proof</u>.
If Y^{-1} and $(Y+X)^{-1}$ exist, we have the matrix identity

$$(Y + X)^{-1} = Y^{-1}(I-X(Y+X)^{-1})$$

(because $(Y+X)\ (Y+X)^{-1} = I \Rightarrow Y(Y+X)^{-1} = I-X(Y+X)^{-1}$).

Using this with $X := B'L(t+1)B$ and $Y := \sum_{i=1}^{N} \alpha_i R_i(t)$, we can write for the P.O.-solution (2):

$$\tilde{u}(t) = -(\sum_{i=1}^{N} \alpha_i R_i(t))^{-1}(I-B'L(t+1)B(\sum_{i=1}^{N} \alpha_i R_i(t)+B'L(t+1)B)^{-1})B'.$$

$$.(L(t+1)((I+A)\tilde{x}(t)+B\,\hat{u}(t)+Cz(t))+f(t+1))+\hat{u}(t) =$$

$$= -(\sum_{i=1}^{N} \alpha_i R_i(t))^{-1}B'(I-L(t+1)B(\sum_{i=1}^{N} \alpha_i R_i(t)+B'L(t+1)B)^{-1}B').$$

$$.(L(t+1)((I+A)\tilde{x}(t)+B\hat{u}(t)+Cz(t))+f(t+1))+\hat{u}(t) =$$

$$= -\left(\sum_{i=1}^{N} \alpha_i R_i(t)\right)^{-1} B'D^{-1}(t+1)'.$$

$$.\left(L(t+1)\left((I+A)\tilde{x}(t)+B\hat{u}(t)+Cz(t)\right)+f(t+1)\right)+\hat{u}(t).$$

It is easy to see, that

$$D^{-1}(t+1)'L(t+1) = L(t+1)D^{-1}(t+1),\ t = t_0,t_0+1,\ldots,t_f-1.$$

Furthermore,
if $R_{ki}(t) \equiv 0,\ i \neq k,\ i = 1,2,\ldots,N,\ \ k = 1,2,\ldots, N,\ t = t_0,t_0+1,\ldots,t_f-1,$

then

$$\sum_{i=1}^{N} \alpha_i R_i(t) = \begin{bmatrix} \alpha_1 R_{11}(t) & & & 0 \\ & \alpha_2 R_{22}(t) & & \\ & & \ddots & \\ 0 & & & \alpha_N R_{NN}(t) \end{bmatrix}, t=t_0,t_0+1,\ldots,t_f-1.$$

So, we can write for the P.O.-control law:

$$\tilde{u}(t) = \begin{bmatrix} \tilde{u}_1(t) \\ \tilde{u}_2(t) \\ \vdots \\ \tilde{u}_N(t) \end{bmatrix} = - \begin{bmatrix} \frac{1}{\alpha_1} R_{11}^{-1}(t) & & & 0 \\ & \frac{1}{\alpha_2} R_{22}^{-1}(t) & & \\ & & \ddots & \\ 0 & & & \frac{1}{\alpha_N} R_{NN}^{-1}(t) \end{bmatrix} \begin{bmatrix} B_1' \\ B_2' \\ \vdots \\ B_N' \end{bmatrix}.$$

$$.\left(L(t+1)D^{-1}(t+1)\left((I+A)\tilde{x}(t)+B\hat{u}(t)+Cz(t)\right) + \right.$$

$$\left. +D^{-1}(t+1)'f(t+1)\right) + [\,\hat{u}_1(t)\ \hat{u}_2(t)\ldots\hat{u}_N(t)]'\ ,$$

$$t = t_0,t_0+1,\ldots,t_f-1,$$

Q.E.D.

4. Incentives to Cooperate.*)

We claimed that whenever the Nash optimal solution does not belong to the set of Pareto optimal solutions at least one player is willing to coordinate the policies. From this it is natural to ask ourselves the question: when does the N.O.-solution belong to the set of P.O.-solutions and when it does not?

In this paragraph we make the assumption:

$R_{ki}(t) \equiv 0,\ i = 1,2,\dots,N,\ k = 1,2,\dots,N,\ t = t_0, t_0+1,\dots,t_f-1.$

First we restate what we found in a previous paper**) and in the previous paragraph:

- N.O.-solution:

$$(14a)\quad u^*_k(t) = -R_{kk}^{-1}(t)B_k'K_k(t+1)E^{-1}(t+1)\Big[(I+A)x^*(t) - \sum_{i=1}^{N} B_i R_{ii}^{-1}(t)B_i'g_i(t+1) + \sum_{i=1}^{N} B_i \hat{u}_i(t) + Cz(t)\Big] - R_{kk}^{-1}(t)B_k'g_k(t+1) + \hat{u}_k(t),$$

$$t = t_0, t_0+1,\dots,t_f-1,\quad k = 1,2,\dots,N,$$

where $E(t+1) := I + \sum_{i=1}^{N} B_i R_{ii}^{-1}(t)B_i'K_i(t+1)$

or, equivalently,

$$(14b)\quad u^*_k(t) = -R_{kk}^{-1}(t)B_k'\big(K_k(t+1)x^*(t+1) + g_k(t+1)\big) + \hat{u}_k(t),$$

$$t = t_0, t_0+1,\dots,t_f-1,\quad k = 1,2,\dots,N.$$

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*) Although in an entirely different way, incentives to cooperate among players in linear-quadratic difference games are also analysed by De Bruyne, G., (1976), Incentives to Cooperation, mimeographed version for a Ph. D.-thesis, Catholic University of Louvain.

**) See Plasmans, J., (1978), op. cit., pp. 24 and ff. or appendix C.

-P.O.-solutions:

for every vector $\alpha \in \mathbb{R}^N$ with $\alpha_k > 0$, $k = 1,2,\ldots,N$, and $\sum_{i=1}^{N} \alpha_k = 1$:

$$(15a)\quad \tilde{u}_k(t) = -\frac{1}{\alpha_k} R_{kk}^{-1}(t)B_k'L(t+1)D^{-1}(t+1)((I+A)\tilde{x}(t)+B\hat{u}(t)+Cz(t)) +$$

$$-\frac{1}{\alpha_k} R_{kk}^{-1}(t)B_k'D^{-1}(t+1)'f(t+1)+\hat{u}_k(t),$$

$$t = t_0,t_0+1,\ldots,t_f-1,\quad k = 1,2,\ldots,N.$$

It is easy to see, starting from equation (2), using the state equation, or from proof B of theorem 1, that relationship (15a) is equivalent to:

$$(15b)\quad \tilde{u}_k(t) = -\frac{1}{\alpha_k} R_{kk}^{-1}(t)B_k'(L(t+1)\tilde{x}(t+1)+f(t+1))+\hat{u}_k(t),$$

$$t = t_0,t_0+1,\ldots,t_f-1,\quad k = 1,2,\ldots,N.$$

- Backward recursive Riccati equations:
  for the closed loop N.O.-solution:

$$(16a)\quad K_k(t_f) = F'Q_k(t_f)F,\quad k = 1,2,\ldots,N;$$

$$(16b)\quad K_k(t) = F'Q_k(t)F + (I+A)'E^{-1}(t+1)'K_k(t+1)\,.$$

$$.(I+B_k R_{kk}^{-1}(t)B_k'K_k(t+1))E^{-1}(t+1)(I+A),$$

$$t = t_0,t_0+1,\ldots,t_f-1,\quad k = 1,2,\ldots,N;$$

for the P.O.-solutions:

$$(17a)\quad L(t_f) = F'\sum_{i=1}^{N} \alpha_i\, Q_i(t_f)F\ ;$$

(17b) $L(t) = F' \sum_{i=1}^{N} \alpha_i Q_i(t)F + (I+A)'L(t+1)D^{-1}(t+1)(I+A)$

$$t = t_0, t_0+1, \ldots, t_f-1.$$

- Backward recursive tracking equations:
  for the closed loop N.O.-solution :

(18a) $g_k(t_f) = -F'Q_k(t_f)F\hat{x}_k(t_f), \quad k = 1,2,\ldots,N;$

(18b) $g_k(t) = -F'Q_k(t)F\hat{x}_k(t) + (I+A)' E^{-1}(t+1)' \,.$

$$.(I+K_k(t+1) B_k R_{kk}^{-1}(t) B_k').$$

$$.(g_k(t+1)+K_k(t+1)E^{-1}(t+1)(\sum_{i=1}^{N} B_i\hat{u}_i(t)+Cz(t)-\sum_{i=1}^{N} B_iR_{ii}^{-1}(t)B_i'g_i(t+1))),$$

$$t = t_0, t_0+1, \ldots, t_f-1;$$

for the P.O.-solutions:

(19a) $f(t_f) = -\sum_{i=1}^{N} F' \alpha_i Q_i(t_f) F \hat{x}_i(t_f);$

(19b) $f(t) = -F' \sum_{i=1}^{N} \alpha_i Q_i(t) F \hat{x}_i(t) + (I+A)' D^{-1}(t+1)'.$

$$.(L(t+1)(B\hat{u}(t)+C z(t))+f(t+1)), \quad t = t_0, t_0+1, \ldots, t_f-1.$$

Before we give sufficient conditions for the closed loop N.O.-solution to be P.O. (in theorem 2), we first prove some lemmas.

<u>Lemma 3</u>.
For all $t \in \{t_0+1, \ldots, t_f\}$ we have:

if $B_k'K_k(t) = \frac{1}{\alpha_k} B_k'L(t), \quad k = 1,2,\ldots,N$, then $E(t) = D(t)$.

Proof.

$$D^{-1}(t+1) = I-B(\sum_{i=1}^{N} \alpha_i R_i(t) + B'L(t+1)B)^{-1} B'L(t+1).$$

when $(I+ST')^{-1}$ and $(I+T'S)^{-1}$ exist, we have the matrix identity:

$$(I + ST')^{-1} = I - S(I + T'S)^{-1} T'$$

(because $(I + ST')(I - S(I + T'S)^{-1}T') =$

$$= I + ST' - S(I + T'S)^{-1}T' - ST'S(I + T'S)^{-1}T' =$$

$$= I + ST' - S((I + T'S)^{-1} + T'S(I + T'S)^{-1})T' = I).$$

Hence, with $S: = B$ and $T': = (\sum_{i=1}^{N} \alpha_i R_i(t))^{-1}B'L(t+1)$, we have:

$$(I+B(\sum_{i=1}^{N} \alpha_i R_i(t))^{-1} B'L(t+1))^{-1} = I-B(I+(\sum_{i=1}^{N} \alpha_i R_i(t))^{-1} B'L(t+1)B)^{-1} .$$

$$.(\sum_{i=1}^{N} \alpha_i R_i(t))^{-1} B'L(t+1) = D^{-1}(t+1),$$

or

$$D(t+1) = I+B(\sum_{i=1}^{N} \alpha_i R_i(t))^{-1} B'L(t+1) =$$

$$= I+\sum_{i=1}^{N} \frac{1}{\alpha_i} B_i R_{ii}^{-1}(t) B_i'L(t+1),$$

so that from the assumptions of this lemma:

$$D(t+1) = I + \sum_{i=1}^{N} B_i R_{ii}^{-1}(t) B_i'K_i(t+1) = E(t+1), \quad t = t_0, t_0+1, \ldots, t_f-1.$$

Q.E.D.

Lemma 4

For all $t \in \{t_0+1,\dots,t_f\}$ we have:
if $B_i'K_j(t) = 0$, $i \neq j$, $i = 1,2,\dots,N$, $j = 1,2,\dots,N$, then

a) $K_k(t)E(t) = K_k(t)(I+B_k R_{kk}^{-1}(t-1)B_k'K_k(t)) = E'(t)K_k(t)$, $k = 1,2,\dots,N$;

b) $K_k(t)E^{-1}(t) = E^{-1}(t)' K_k(t)$, $k = 1,2,\dots,N$.

Proof.

a) $$K_k(t)\ E(t) = K_k(t)(I+\sum_{i=1}^{N} B_i R_{ii}^{-1}(t-1)B_i'K_i(t)) =$$

$$= K_k(t)(I+B_k R_{kk}^{-1}(t-1)B_k'K_k(t)) =$$

$$= (I+K_k(t)B_k R_{kk}^{-1}(t-1)B_k')K_k(t) = E'(t)K_k(t), \quad k = 1,2,\dots,N.$$

b) $$E^{-1}(t)'\ K_k(t) = E^{-1}(t)'\ K_k(t)E(t)E^{-1}(t) =$$

$$\underset{\substack{\uparrow\\(a)}}{=} E^{-1}(t)'E'(t)\ K_k(t)E^{-1}(t) = K_k(t)E^{-1}(t), \quad k = 1,2,\dots,N.$$

Q.E.D.

Corollary 3.

If $B_i'\ K_j(t) = 0$, $i \neq j$, $i = 1,2,\dots,N$, $j = 1,2,\dots,N$, $t = t_0+1,\dots,t_f$, then the matrix-Riccati and tracking equations (16b) and (18b) reduce to:

(16b') $$K_k(t) = F'Q_k(t)F + (I+A)'E^{-1}(t+1)'K_k(t+1)(I+A),$$

$$t = t_0,\ t_0+1,\dots,t_f-1, \quad k = 1,2,\dots,N.$$

$$(18b')\ g_k(t) = -F'Q_k(t)F\,\hat{x}_k(t) + (I+A)'E^{-1}(t+1)'\,\cdot$$

$$\cdot(K_k(t+1)(B\,\hat{u}(t) + C\,z(t)) + g_k(t+1)),$$

$$t = t_0, t_0+1, \ldots, t_f-1, \quad k = 1,2,\ldots,N.$$

Proof.

(16b'): immediately clear from lemma 4a.

(18b'):

$$(I+A)'E^{-1}(t+1)'(I+K_k(t+1)\,B_k\,R_{kk}^{-1}(t)\,B_k')\,\cdot$$

$$\cdot(g_k(t+1)+K_k(t+1)E^{-1}(t+1))(B\,\hat{u}(t) + C\,z(t) - \sum_{i=1}^{N} B_i R_{ii}^{-1}(t) B_i' g_i(t+1)) =$$

$$\underset{\substack{\uparrow \\ \text{lemma 4a}}}{=} (I+A)'E^{-1}(t+1)'((I+K_k(t+1)B_k\,R_{kk}^{-1}(t)B_k')\,g_k(t+1) +$$

$$+ K_k(t+1)(B\,\hat{u}(t)+C\,z(t) - \sum_{i=1}^{N} B_i R_{ii}^{-1}(t)B_i'g_i(t+1))) =$$

$$\underset{\substack{\uparrow \\ \text{assumption}}}{=} (I+A)'E^{-1}(t+1)'(g_k(t+1)+K_k(t+1)(B\,\hat{u}(t)+C\,z(t))),$$

$$t = t_0, t_0+1, \ldots, t_f-1, \quad k = 1,2,\ldots,N.$$

Q.E.D.

Lemma 5.

$B_k'\,K_k(t) = \frac{1}{\alpha_k} B_k'L(t)$, $t = t_0+1,\ldots,t_f$, $k = 1,2,\ldots,N$, if and only if

a) $L(t) = \sum_{i=1}^{N} \alpha_i\,K_i(t)$, $t = t_0+1,\ldots,t_f$, and

b) $B_i'K_j(t) = 0$, $t = t_0+1,\ldots t_f$, $i \neq j$, $i = 1,2,\ldots,N$, $j = 1,2,\ldots,N$.

Proof.

(sufficiency)

$$\frac{1}{\alpha_k} B_k{}'L(t) \underset{\uparrow \atop (a)}{=} \frac{1}{\alpha_k} B_k{}' \sum_{i=1}^{N} \alpha_i K_i(t) \underset{\uparrow \atop (b)}{=} B_k{}'K_k(t),$$

$$t = t_0+1,\ldots,t_f, \quad k = 1,2,\ldots,N.$$

(necessity)

We will proof this by induction:

(i) $t = t_f$: a) $L(t_f) = \sum_{i=1}^{N} \alpha_i K_i(t_f)$ from (16a) and (17a)

b) We know that for all $i \in \{1,2,\ldots,N\}$

$$\alpha_i B_i{}' K_i(t_f) = B_i{}'L(t_f) \underset{\uparrow \atop (a)}{=} B_i{}' \sum_{j=1}^{N} \alpha_j K_j(t_f) \Rightarrow$$

$$\Rightarrow B_i{}' \sum_{\substack{j=1 \\ j\neq i}}^{N} \alpha_j K_j(t_f) = 0 \Rightarrow B_i{}' \sum_{\substack{j=1 \\ j\neq i}}^{N} \alpha_j K_j(t_f) B_i = 0.$$

Because $\alpha_j > 0$, $j = 1,2,\ldots,N$ and $K_j(t_f)$ positive semi-definite, $j = 1,2,\ldots,N$, we can conclude, that for all columns b_{ik}, $k = 1,2,\ldots,s_i$, of B_i, for all $i = 1,2,\ldots,N$, $i \neq j$,

$$b_{ik}' K_j(t_f) b_{ik} = 0,$$

and since $K_j(t_f)$, $j = 1,2,\ldots,N$, is both symmetric and positive semi-definite, we can split this matrix as $K_j(t_f) = T_j(t_f)T_j{}'(t_f)$, $j = 1,2,\ldots,N$, where $T_j(t_f)$, $j = 1,2,\ldots,N$, are lower triangular matrices with non-negative elements on the diagonal.

Hence, we have:

$$b_{ik}{}'T_j(t_f)T_j'(t_f)b_{ik} = 0 \quad \text{for all } k \in \{1,2,\ldots,s_i\}\ ,\quad i \neq j,$$

$$i \in \{1,2,\ldots,N\},\ j \in \{1,2,\ldots,N\}\ ,$$

so that $b_{ik}{}'\,T_j(t_f) = 0 \Rightarrow$

$$\Rightarrow b_{ik}{}'\,T_j(t_f)T_j{}'(t_f) = 0 \Rightarrow$$

$$\Rightarrow b_{ik}{}'\,K_j(t_f) = 0 \quad \text{for all } k \in \{1,2,\ldots,s_i\} \Rightarrow$$

$$\Rightarrow B_i{}'\,K_j(t_f) = 0 \quad \text{for all } i \neq j,\ \ i\in\{1,2,\ldots,N\}\ ,\ j\in\{1,2,\ldots,N\}.$$

(ii) Suppose (a) and (b) are true for $t=t',\ t'+1,\ldots,t_f$, then:

a) $$L(t'-1) = F'\sum_{i=1}^{N}\alpha_i\,Q_i(t'-1)F + (I+A)'L(t')D^{-1}(t')(I+A) =$$

(17b)

$$= F'\sum_{i=1}^{N}\alpha_i\,Q_i(t'-1)F + (I+A)'D^{-1}(t')'\,L(t')(I+A) =$$

$$= F'\sum_{i=1}^{N}\alpha_i\,Q_i(t'-1)F + (I+A)'E^{-1}(t')'\,L(t')E(t')E^{-1}(t')(I+A) =$$

(lemma 3)

$$= F'\sum_{i=1}^{N}\alpha_i\,Q_i(t'-1)F + (I+A)'E^{-1}(t')'\sum_{i=1}^{N}\alpha_i\,K_i(t')E(t')E^{-1}(t')(I+A) =$$

induction
assumption(a)

$$= F'\sum_{i=1}^{N}\alpha_i\,Q_i(t'-1)F + \sum_{i=1}^{N}\alpha_i(I+A)'E^{-1}(t')'K_i(t').$$

induction
assumption(b)
and
lemma 4a

$$.(I+B_i\,R_{ii}{}^{-1}(t'-1)B_i{}'K_i(t'))E^{-1}(t')(I+A) =$$

$$= \sum_{i=1}^{N} \alpha_i \, K_i(t'-1).$$

(16b)

b) Here we can follow exactly the same reasoning as in (i), part (b), except for $t = t'-1$ instead of $t = t_f$.

Q.E.D.

Lemma 6.

If $B_k' K_k(t) = \frac{1}{\alpha_k} B_k' L(t)$, $t = t_0+1,\ldots,t_f$, $k = 1,2,\ldots,N$, then

$$B_k' g_k(t) = \frac{1}{\alpha_k} B_k' f(t), \quad t = t_0+1,\ldots,t_f, \quad k = 1,2,\ldots,N.$$

Proof.

step 1: $f(t) = \sum_{i=1}^{N} \alpha_i \, g_i(t)$, $t = t_0+1,\ldots,t_f$.

proof:

We prove this by induction:

(i) $t = t_f$: immediately clear from (18a) and (19a).

(ii) suppose it is true for $t = t', t'+1,\ldots,t_f$, then it is easy to see from (18b') and (19b), using lemma 3 and lemma 5a, that it is also true for $t = t'-1$.

step 2: $B_i' g_j(t) = 0$, $i \neq j$, $i = 1,2,\ldots,N$, $j = 1,2,\ldots,N$, $t = t_0+1,\ldots,t_f$.

proof:

From lemma 5b we know

$B_i' K_j(t) = 0$, $i \neq j$, $i = 1,2,\ldots,N$, $j = 1,2,\ldots,N$, $t = t_0+1,\ldots,t_f$.

Using (16a) and (16b') we find:

(20) $$B_i' F' Q_j(t_f) F = 0, \quad i \neq j, \quad i = 1,2,\ldots,N, \quad j = 1,2,\ldots,N \text{ and}$$

$$B_i' F' Q_j(t) F + B_i'(I+A)' E^{-1}(t+1)' K_j(t+1)(I+A) = 0,$$

$$t = t_0+1,\ldots,t_f-1, \quad i \neq j, \quad i = 1,2,\ldots,N, \quad j = 1,2,\ldots,N.$$

Furthermore, we know from the fact that $K_j(t+1)$ is symmetric and positive semi-definite, from the model assumptions and from lemma 4b, that both $Q_j(t)$ and $E^{-1}(t+1)'K_j(t+1)$ are symmetric and positive semi-definite. By the same reasoning as used in proving lemma 5b, where B_i' and $F'Q_j(t)F$, respectively $B_i'(I+A)'$ and $E^{-1}(t+1)'K_j(t+1)$, play the rôle of B_i' and $K_j(t_f)$, we can conclude that:

(21) $B_i'F'Q_j(t)F = 0,\ i \neq j,\ i = 1,2,\ldots,N,\ j = 1,2,\ldots,N,\ t = t_0+1,\ldots,t_f-1,$

and

(22) $$B_i'(I+A)'E^{-1}(t+1)'K_j(t+1) \underset{\substack{\uparrow \\ \text{lemma 4b}}}{=} B_i'(I+A)'K_j(t+1)E^{-1}(t+1) = 0,$$

$$i \neq j,\ i = 1,2,\ldots,N,\ j = 1,2,\ldots,N,\ t = t_0+1,\ldots,t_f-1.$$

Postmultiplication of both sides of equation (22) with $E(t+1)$ yields:

(23) $B_i'(I+A)'K_j(t+1) = 0,\ i \neq j,\ i = 1,2,\ldots,N,\ j = 1,2,\ldots,N,\ t = t_0+1,\ldots,t_f-1.$

Now we will evaluate $E^{-1}(t+1)'$, $t = t_0,t_0+1,\ldots,t_f-1$:

$$E^{-1}(t+1)' \underset{\substack{\uparrow \\ \text{lemma 3}}}{=} D^{-1}(t+1)' \underset{\substack{\uparrow \\ \text{definition}}}{=} I-L(t+1)B(\sum_{i=1}^{N} \alpha_i R_i(t)+B'L(t+1)B)^{-1} B' .$$

Because we assumed $R_{ij}(t) \equiv 0,\ i \neq j$, and because $B_i'K_j(t+1) = 0,\ i \neq j$, and $B_k'K_k(t+1)= \frac{1}{\alpha_k} B_k'L(t+1)$, $i = 1,2,\ldots,N,\ j = 1,2,\ldots,N,\ k = 1,2,\ldots,N$, we can write:

$$M(t) := \sum_{i=1}^{N} \alpha_i R_i(t) + B'L(t+1)B = \begin{bmatrix} \alpha_1 R_{11}(t)+B_1'\alpha_1 K_1(t+1)B_1 & & O \\ & \ddots & \\ O & & \alpha_N R_{NN}(t)+B_N'\alpha_N K_N(t+1)B_N \end{bmatrix}$$

It follows that

$$E^{-1}(t+1)' = I - [\alpha_1 K_1(t+1)B_1 \ldots \alpha_N K_N(t+1)B_N] M^{-1}(t) [B_1 \ldots B_N]' =$$

$$= I - \sum_{i=1}^{N} K_i(t+1)B_i(R_{ii}(t)+B_i'K_i(t+1)B_i)^{-1} B_i' .$$

Now we are ready to prove step 2.
We will do this again by induction:

(i) $t = t_f$: obvious from (18a) and (20).

(ii) Suppose it is true for $t = t', t'+1, \ldots, t_f$.
Starting from (18b'), using (21) and (22), we find:

$$B_i'g_j(t'-1) = B_i'(I+A)'E^{-1}(t')'g_j(t'), \quad i \neq j, \ i = 1,2,\ldots,N, \ j = 1,2,\ldots,N.$$

Taking account of (23) and the induction assumption $B_i'g_j(t') = 0$, after substituting (24), we are left with:

$$B_i'g_j(t'-1) = B_i'(I+A)'g_j(t'), \quad i \neq j, \ i = 1,2,\ldots,N, \ j = 1,2,\ldots,N.$$

Now consider (23) again:

$$B_i'(I+A)'K_j(t) = 0, \quad i \neq j, \ i = 1,2,\ldots,N, \ j = 1,2,\ldots,N, \ t = t_0+2,\ldots,t_f.$$

With $B_i'(I+A)'$ in the rôle of B_i' we have the same situation as in the beginning of the proof of step 2. Following the same reasoning as we did before, we arrive successively at:

(20') $B_i'(I+A)'F'Q_j(t_f)F = 0, \ i \neq j, \ i = 1,2,\ldots,N, \ j = 1,2,\ldots,N.$

(21') $B_i'(I+A)'F'Q_j(t)F = 0, \ i \neq j, \ i = 1,2,\ldots,N, \ j=1,2,\ldots,N,$
$t=t_0+2,\ldots,t_f-1.$

(22') $B_i'(I+A)'^2 E^{-1}(t+1)' K_j(t+1) = 0 \quad i\neq j,\ i=1,2,\dots,N,\ j=1,2,\dots,N$

(23') $B_i'(I+A)'^2 K_j(t+1) = 0$, $i\neq j,\ i=1,2,\dots,N,\ j=1,2,\dots,N,\ t=t_0+2,\dots,t_f-1,$

and, with induction assumption $B_i'g_j(t'+1) = 0$ this time,

$$\begin{aligned} B_i'g_j(t'-1) &= B_i'(I+A)'g_j(t') = \\ &= B_i'(I+A)'^2 E^{-1}(t'+1)'g_j(t'+1) = \\ &= B_i'(I+A)'^2 g_j(t'+1), i\neq j, i=1,2,\dots,N, j=1,2,\dots,N. \end{aligned}$$

Continuing this process we will finally find:

$$\begin{aligned} B_i'g_j(t'-1) &= B_i'(I+A)'^{(t_f-t'+1)}g_j(t_f) = \\ &= -B_i'(I+A)'^{(t_f-t'+1)}K_j(t_f)\hat{x}_j(t_f),\ i\neq j, i=1,2,\dots,N, j=1,2,\dots,N; \end{aligned}$$

and also

$$B_i'(I+A)'^{(t_f-t'+1)}K_j(t_f) = 0,\ i\neq j,\ i = 1,2,\dots,N,\ j = 1,2,\dots,N,$$

so that

$$B_i'g_j(t'-1) = 0,\ i \neq j,\ i = 1,2,\dots,N,\ j = 1,2,\dots,N.$$

(i) + (ii): $B_i'g_j(t) = 0,\ i\neq j,\ i = 1,2,\dots,N,\ j = 1,2,\dots,N,\ t = t_0+1,\dots,t_f.$

step 3: for all $t \in \{t_0+1,\dots,t_f\}$ and $k \in \{1,2,\dots,N\}$ we have:

$$\frac{1}{\alpha_k} B_k'f(t) \underset{\substack{\uparrow \\ \text{step 1}}}{=} \frac{1}{\alpha_k} B_k' \sum_{i=1}^{N} \alpha_i g_i(t) \underset{\substack{\uparrow \\ \text{step 2}}}{=} \frac{1}{\alpha_k} B_k'\alpha_k g_k(t) = B_k'g_k(t) .$$

Q.E.D.

By now we have almost proved the main result of this paragraph, namely sufficient conditions for the (closed-loop) N.O.-solution to be P.O.. We will state this result in the following theorem.

Theorem 2.

The (closed-loop) N.O.-solution of the linear quadratic difference game is P.O. if for some vector $\alpha \in \mathbb{R}^N$, with $\alpha_k > 0$, $k = 1,2,\ldots,N$, and $\sum_{k=1}^{N} \alpha_k = 1$,

$$B_k' K_k(t) = \frac{1}{\alpha_k} B_k' L(t), \quad k=1,2,\ldots,N, \quad t=t_0+1,\ldots,t_f,$$

where $K_k(t)$ and $L(t)$ are the solutions of the backward recursive Riccati equations (16a), (16b) resp. (17a),(17b).

Proof.

We will prove this by induction, comparing (14a) and (15a):

(i) $t = t_0$: We know $x^*(t_0) = \tilde{x}(t_0) = x_0$.
By lemma 3 we know $E(t_0+1) = D(t_0+1)$.
Furthermore, for $k = 1,2,\ldots,N$:

$$R_{kk}^{-1}(t_0)B_k'K_k(t_0+1)E^{-1}(t_0+1)\sum_{i=1}^{N} B_i R_{ii}^{-1}(t_0)B_i'g_i(t_0+1) +$$

$$-R_{kk}^{-1}(t_0)B_k'g_k(t_0+1) =$$

$$\underset{\text{lemmas 4b, 5b}}{=} R_{kk}^{-1}(t_0)B_k'E^{-1}(t_0+1)'K_k(t_0+1)\sum_{i=1}^{N} B_i R_{ii}^{-1}(t_0)B_i'g_i(t_0+1) +$$

$$-R_{kk}^{-1}(t_0)B_k'g_k(t_0+1) =$$

$$\underset{\text{lemma 5b}}{=} -R_{kk}^{-1}(t_0)B_k'(I-E^{-1}(t_0+1)'K_k(t_0+1)B_kR_{kk}^{-1}(t_0)B_k')g_k(t_0+1) =$$

$$\underset{\text{lemma 6}}{=} \frac{1}{\alpha_k} R_{kk}^{-1}(t_0)B_k'(I-E^{-1}(t_0+1)'K_k(t_0+1)B_kR_{kk}^{-1}(t_0)B_k')f(t_0+1) =$$

$$= -\frac{1}{\alpha_k} R_{kk}^{-1}(t_0)B_k'(I-K_k(t_0+1)E^{-1}(t_0+1)B_kR_{kk}^{-1}(t_0)B_k')f(t_0+1) =$$

$\uparrow$
lemmas 4b, 5b

$$= -\frac{1}{\alpha_k} R_{kk}^{-1}(t_0)B_k'(I-\sum_{i=1}^{N} K_i(t_0+1)E^{-1}(t_0+1)B_iR_{ii}^{-1}(t_0)B_i')f(t_0+1) =$$

$\uparrow$
lemma 5b

$$= -\frac{1}{\alpha_k} R_{kk}^{-1}(t_0)B_k'(I-E^{-1}(t_0+1)'\sum_{i=1}^{N} K_i(t_0+1)B_iR_{ii}^{-1}(t_0)B_i')f(t_0+1) =$$

$\uparrow$
lemmas 4b, 5b

$$= -\frac{1}{\alpha_k} R_{kk}^{-1}(t_0)B_k'E^{-1}(t_0+1)'(E'(t_0+1)-\sum_{i=1}^{N} K_i(t_0+1)B_iR_{ii}^{-1}(t_0)B_i')f(t_0+1)=$$

$$= -\frac{1}{\alpha_k} R_{kk}^{-1}(t_0)B_k'E^{-1}(t_0+1)'f(t_0+1) =$$

$\uparrow$
definition

$$= -\frac{1}{\alpha_k} R_{kk}^{-1}(t_0)B_k'D^{-1}(t_0+1)'f(t_0+1).$$

$\uparrow$
lemma 3

Hence, we can conclude $u^*_k(t_0) = \tilde{u}_k(t_0)$, $k = 1,2,\dots,N$, for this vector $\alpha \in \mathbb{R}^N$.

(ii) Suppose $u^*_k(t) = \tilde{u}_k(t)$ for $t = t_0, t_0+1, \dots, t'$, $k = 1,2,\dots,N$, for this vector $\alpha \in \mathbb{R}^N$, then also

$$x^*(t) = \tilde{x}(t) \text{ for } t = t_0, t_0+1, \dots, t'+1.$$

The same reasoning as in (i) gives us:

$$u^*_k(t'+1) = \tilde{u}_k(t'+1),\quad k = 1,2,\dots,N, \text{ for this vector } \alpha \in \mathbb{R}^N.$$

Q.E.D.

The conditions are not necessary.
For example, suppose that the matrices (I+A), B and C have zeros on their first rows and suppose that the N.O.-solution is P.O. or,

equivalently,

$u^*_k(t) = \tilde{u}_k(t)$, $k = 1,2,\ldots,N$, $t = t_0,t_0+1,\ldots,t_f-1$, for some vector $\hat{\alpha} \in \mathbb{R}^N$, with $\hat{\alpha}_k > 0$, $k = 1,2,\ldots,N$, and $\sum_{k=1}^{N} \hat{\alpha}_k = 1$, then also $x^*(t) = \tilde{x}(t)$, $t = t_0,t_0+1,\ldots,t_f$, and $x^*(t)$, $t = t_0+1,\ldots,t_f$, has zero as its first element. Comparing (14b) and (15b) we see that this situation is very well compatible with a situation where $\frac{1}{\alpha_k} B_k'L(t)$ and $B_k'K_k(t)$, $k = 1,2,\ldots,N$, $t = t_0+1,\ldots,t_f$, differ in their first column for all vectors $\alpha \in \mathbb{R}^N$, with $\alpha_k > 0$, $k = 1,2,\ldots,N$, and $\sum_{k=1}^{N} \alpha_k = 1$.

5. Conclusion.

In the last paragraph of this paper, we have derived a sufficient condition for a (closed-loop) N.O.-strategy to be P.O. (under the assumption that $R_{ij}(t) \equiv 0$ for $i \neq j$), which actually means that in such a case not all the players have an incentive to cooperate.

Given a quadratic, strict convex cost functional and assuming that the dynamic behaviour of the economy is described by a system of linear difference equations as stated under problem 1 on page 13, then there will be no incentive to cooperate if the condition of theorem 2 is fulfilled. What does this condition mean?

Considering $\frac{\partial J_i}{\partial u_j(t)}$ $(i \neq j)$ and taking account of (20), (20'), (21'), etc., we find:

$$\frac{\partial J_i}{\partial u_j(t)} = \sum_{\tau=1}^{t_f-t} \left(\frac{\partial x(t+\tau)}{\partial u_j(t)}\right)' \frac{\partial J_i}{\partial x(t+\tau)} =$$

$$= \sum_{\tau=1}^{t_f-t} B_j'(I+A)'^{\tau-1} F'Q_i(t+\tau)F[x(t+\tau)-\hat{x}_i(t+\tau)] = 0$$

$$(i=1,2,\ldots,N;\ j=1,2,\ldots,N,\ i\neq j,\ t=t_0,t_0+1,\ldots,t_f-1),$$

so that the sufficient condition implies that the cost functional of any one player is not influenced by the strategies of the other players.
This is, obviously, a very stringent condition, but, fortunately, it is not necessary for a N.O.-solution to be P.O.

Therefore, we consider paragraph 3, where the set of all P.O.-strategies for the linear-quadratic difference game and the corresponding P.O.-costs are derived, as the main part of this paper. In practice, it will be useful to compute the N.O.-strategy and the set of P.O.-strategies for each player, so that both types of strategies can be compared. The policy-maker has to make a trade-off between the N.O.- and the P.O.-costs, in order to decide for cooperation or not.
Interesting could be to consider randomized decisions.

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